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Front-Form Hamiltonian, Path Integral, and BRST Formulations of the Nonlinear Sigma Model

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The nonlinear sigma model in one-space one-time dimension is considered on the light-front. The front-form theory is seen to possess a set of three first-class constraints and consequently it possesses a local vector gauge symmetry. This is in contrast to the usual instant-form theory, which is well known to be a gauge noninvariant theory possessing a set of four second-class constraints. The front-form Hamiltonian, path integral, and BRST formulations of this theory are investigated under some specific gauge choices.

KEY WORDS: front form Hamiltonian; path integral; nonlinear sigma model.

1. INTRODUCTION

The O(N) nonlinear sigma models (NLSM) in one-space one-time ((1+1)-) dimension (Callen *et al.*, 1969; Candelas *et al.*, 1985; Coleman *et al.*, 1969; Henneaux and Mezincescu, 1985; Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b; Ruehl, 1991a,b, 1993, 1995, 1996; Zamolodchikov and Zamolodchikov, 1979), where the field sigma is a real N-component field, provide a laboratory for the various nonperturbative techniques, e.g., $1/N$-expansion (Ruehl, 1991a,b, 1993, 1995, 1996), operator product expansion, and the low energy theorems (Callen *et al.*, 1969; Coleman *et al.*, 1969). These models are characterized by features like renormalization and asymptotic freedom common with that of quantum chromodynamics and exhibit a nonperturbative particle spectrum, have no intrinsic scale parameter, possess topological charges, and are very crucial in the context of conformal (Ruehl, 1991a,b, 1993, 1995, 1996) and string field theories (Candelas *et al.*, 1985; Henneaux and Mezincescu, 1985;) where they appear in the classical limit (Callen *et al.*, 1969; Coleman *et al.*, 1969).

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The Hamiltonian formulation of the gauge-noninvariant (GNI) O(N)-NLSM in (1+1)-dimension has been studied in Maharana (1983a) and its two gauge-invariant (GI) versions have been constructed in Kulshreshtha *et al.* (1993a), where the Hamiltonian (Dirac, 1950, 1964) and Becchi–Rouet–Stora and Tyutin (BRST) (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1998; Tyutini, 1975) quantization of these GI models have also been studied in detail. In the present work, we propose to investigate the canonical structure, constrained dynamics, and Hamiltonian (Dirac, 1950, 1964), path integral, and BRST (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998;Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1998; Tyutini, 1975) formulations of this model on the light-front (LF), i.e., on the hyperplanes: light-cone (LC) time $x^+ \equiv t = (x^0 + x^1)/\sqrt{2} =$ constant (Dirac, 1949; for a recent review see, e.g., Brodsky *et al.* 1998). The Hamiltonian and BRST formulations of this NLSM in the usual instant form (IF) of dynamics (on the hyperplanes $x^0 =$ constant) (Dirac, 1949; for a recent review see, e.g., Brodsky *et al.*, 1998) have been investigated in Kulshreshtha *et al.* (1993). The IF theory (Kulshreshtha *et al.* 1993a) is well known to be a GNI theory possessing a set of four second-class constraints (Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b; Kulshreshtha *et al.*, 1993a). On the other hand, the front-form (FF) theory under the present investigation is seen to possess a set of three first-class constraints, and consequently it describes a GI theory. The FF Hamiltonian and path integral formulation of this model has been investigated under some specific gauges in the present work.

Also, because the LF coordinates are not related to the conventional IF coordinates by a finite Lorentz transformation, the descriptions of the same physical result may be different in the IF and the FF. In fact, the quantization of relativistic field theories at fixed LC time proposed by Dirac (1989; for a recent review see, e.g., Brodsky *et al.*, 1998) has very important applications and the LF variables are very useful not only in field theories but also in the description of string theories and D-brane physics. In the LC quantization (LCQ) of gauge theories the transverse degrees of freedom of the gauge field can be imediately identified as the dynamical degrees of freedom, and as a result, the LCQ remains very economical in displaying the relevant degrees of freedom leading directly to the physical Hilbert space. In the context of LCQ of two-dimensional field theories, it is very often found that a theory that is gauge anomalous in the IF is no longer gauge anomalous (and therefore gauge invariant) in the FF/LCQ, as seen in the present case of NLSM. Also, in the LCQ, there is usually no conflict with the microcausality, which is in contrast with the usual IF quantization. Also, the FF has seven kinematical Poincare generators including the Lorentz boost transformations compared to only six in the usual IF framework. The advantage of the FF/LCQ over that of the conventional IF quantization is best illustrated in a recent review (Brodsky *et al.*, 1998).

However, in the usual Hamiltonian formulation of a GI theory under some gauge-fixing conditions, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under gauge fixing). To achieve the quantization of a GI theory such that the gauge invariance of the theory is maintained even under gauge fixing, one goes to a more generalized procedure called the BRST formulation (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1998; Tyutini, 1975). In the BRST formulation of a GI theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called the BRST symmetry. For this, one enlarges the Hilbert space of the GI theory and replaces the notion of the gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes the operators having different statistics. In view of this, one introduces new anticommuting variables c and $\bar{c}$ called the Faddeev–Popov ghost and antighost fields, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable b called the Nakanishi–Lautrup field (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1998; Tyutini, 1975).

In the BRST formulation of a theory one thus embeds a GI theory into a BRST-invariant system, and the quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator Q as well as with the anti-BRST charge operator $\bar{Q}$. The new symmetry of the system (the BRST symmetry) that replaces the gauge invariance is maintained (even under gauge fixing) and hence projecting any state onto the sector of BRST and anti-BRST invariant states yields a theory that is isomorphic to the original GI theory. The unitarity and consistency of the BRST-invariant theory described by the gauge-fixed quantum Lagrangian is guaranteed by the conservation and nilpotency of the BRST charge Q.

In the next section, we briefly consider the basics of the O(N)-NLSM (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b), in the IF of dynamics (Kulshreshtha *et al.*, 1993a). In Section 3, we study the Hamiltonian and path integral formulations of this model on the LF under gauge fixing and in Section 4, its BRST formulation under some specific LC gauges. The summary and discussions are finally given in Section 5.

2. THE INSTANT-FORM THEORY

The O(N)-NLSM in one-space one-time dimension in the usual IF (i.e., on the hyperplanes $x^0 = \text{constant}$) is described by the action (Callen *et al.*, 1969; Candelas *et al.*, 1985; Coleman *et al.*, 1969; Henneaux and Mezincescu, 1985;

Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b; Ruehl, 1991a,b, 1993, 1995, 1996; Zamolodchikov and Zamolodchikov, 1979):

$$S = \int \mathscr{L}^N \, dx \, dt \tag{2.1a}$$

$$\mathscr{L}^N = \left[\frac{1}{2}\partial_\mu \sigma_k \partial^\mu \sigma_k + \lambda\left(\sigma_k^2 - 1\right)\right] \quad k = 1, 2, \ldots, N \tag{2.1b}$$

$$= \left[\frac{1}{2}\left(\dot{\sigma}_k^2 - \sigma_k'^2\right) + \lambda\left(\sigma_k^2 - 1\right)\right] \quad k = 1, 2, \ldots, N \tag{2.1c}$$

$$g^{\mu\nu} := \text{diag}(+1, -1) \tag{2.1d}$$

Here $\vec{\sigma} \equiv [\sigma_k(x, t); k = 1, 2, \ldots, N]$ is a multiplet of N real scalar fields in (1+1)-dimension and $\lambda(x, t)$ is another scalar field. The overdots and primes denote the time and space derivatives respectively. The field $\vec{\sigma}(x, t)$ maps the two-dimensional space-time into the N-dimensional internal manifold whose coordinates are $\sigma_k(x, t)$. In the above equation, the first term corresponds to a massless boson (which is equivalent to a massless fermion), and the second term is the usual term involving the nonlinear constraint ($\sigma_k^2 - 1 \approx 0$) and the auxiliary field λ. This model is seen to possess a set of four second-class constraints (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b):

$$\rho_1 = p_\lambda \approx 0 \tag{2.2a}$$

$$\rho_2 = \left[\sigma_k^2 - 1\right] \approx 0 \tag{2.2b}$$

$$\rho_3 = 2\sigma_k \Pi_k \approx 0 \tag{2.2c}$$

$$\rho_4 = \left(2\Pi_k^2 + 4\lambda\,\sigma_k^2 + 2\sigma_k\,\sigma_k''\right) \approx 0 \tag{2.2d}$$

where ρ_1 is a primary constraint and ρ_2, ρ_3, and ρ_4 are secondary constraints. Here Π_k and p_λ are the momenta canonically conjugate respectively to σ_k and λ. The nonvanishing equal-time Dirac brackets (DBs) of the theory are given by (Kulshreshtha *et al.*, 1993a; Maharana, 1983; Mitra and Rajaraman, 1990a,b).

$$\{\Pi_\ell(x), \Pi_m(y)\}_{\text{D}} = -\frac{1}{2}[\sigma_\ell(x)\Pi_m(y) - \Pi_\ell(x)\sigma_m(y)]\,\delta(x - y) \tag{2.3a}$$

$$\{\sigma_\ell(x), \Pi_m(y)\}_{\text{D}} = \left[\delta_{\ell m} - \frac{\sigma_\ell(x)\sigma_m(y)}{\sigma_k^2}\right]\delta(x - y) \tag{2.3b}$$

For achieving the canonical quantization of the theory, one encounters the problem of operator ordering while going from DBs to the commutation relations. This problem could, however, be resolved as explained in Maharana (1983a,b) and Kulshreshtha *et al.* (1993a) by demanding that all the fields and field momenta after quantization become hermition operators and that all the canonical commutation relations be consistent with the hermiticity of these operators (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b).

3. THE LIGHT-FRONT THEORY

In order to study the theory on the LF (i.e., on the hyperplanes $x^+ = (x^0 + x^1)/\sqrt{2} =$ constant) one defines the LC coordinates $x^\pm := [(x^0 \pm x^1)/\sqrt{2}]$ and then writes all the quantities involved in the action in terms of $x^\pm$ instead of x^0 and x^1 (Dirac, 1949; for a recent review see. e.g., Brodsky *et al.*, 1998). The action of the theory on the LF thus reads

$$S = \int \mathscr{L}\, dx^+ dx^- \tag{3.1a}$$

$$\mathscr{L} = \left[(\partial_+\sigma_k)(\partial_-\sigma_k) + \lambda\left(\sigma_k^2 - 1\right)\right] \tag{3.1b}$$

$$\partial_\pm\sigma_k = \left(\overset{\circ}{\sigma}_k \pm \sigma_k'\right)/\sqrt{2} \tag{3.1c}$$

As before, in (3.1b), the first term corresponds to a massless boson (which is equivalent to a massless fermion), and the second term is the usual term involving the nonlinear constraint $[(\sigma_k^2 - 1) \approx 0]$ and the auxiliary field λ. The Euler–Lagrange equations obtained from $\mathscr{L}$ (3.1) are

$$[\partial_+\partial_-\sigma_k - \lambda\,\sigma_k] = 0 \tag{3.2a}$$

$$\left[\sigma_k^2 - 1\right] = 0 \tag{3.2b}$$

3.1. The FF Hamiltonian and Path Integral Formulations

The LC canonical momenta for the above NLSM obtained from $\mathscr{L}$ (3.1) are

$$\Pi_k := \frac{\partial \mathscr{L}}{\partial(\partial_+\sigma_k)} = [\partial_-\sigma_k] \tag{3.3a}$$

$$\rho_\lambda := \frac{\partial \mathscr{L}}{\partial(\partial_+\lambda)} = 0 \tag{3.3b}$$

Here, Π_k and p_λ are the momenta canonically conjugate respectively to σ_k and λ. Also the above equations imply that the theory possesses two primary constraints:

$$\chi_1 = p_\lambda \approx 0 \tag{3.4a}$$

$$\chi_2 = [\Pi_k - \partial_-\sigma_k] \approx 0 \tag{3.4b}$$

The canonical Hamiltonian density corresponding to $\mathscr{L}$ is

$$\mathscr{H}_C = [\Pi_k(\partial_+\sigma_k) + p_\lambda(\partial_+\lambda) - \mathscr{L}] = \left[-\lambda\left(\sigma_k^2 - 1\right)\right] \tag{3.5}$$

After including the primary constraints χ_1 and χ_2 in the canonical Hamiltonian density $\mathscr{H}_{\rm C}$ with the help of Lagrange multipliers u and v, one can write the total Hamiltonian density $\mathscr{H}_{\rm T}$ as

$$\mathscr{H}_{\rm T} = \left[-\lambda\left(\sigma_k^2 - 1\right) + p_\lambda u + (\Pi_k - \partial_- \sigma_k)v\right] \tag{3.6}$$

The Hamiltons equations obtained from the total Hamiltonian $H_{\rm T} = \int \mathscr{H}_{\rm T}\, dx^-$ are

$$\partial_+ \sigma_k = \frac{\partial H_{\rm T}}{\partial \Pi_k} = v \tag{3.7a}$$

$$-\partial_+ \Pi_k = \frac{\partial H_{\rm T}}{\partial \sigma_k} = [-2\lambda\, \sigma_k + \partial_- v] \tag{3.7b}$$

$$\partial_+ \lambda = \frac{\partial H_{\rm T}}{\partial p_\lambda} = u \tag{3.7c}$$

$$-\partial_+ p_\lambda = \frac{\partial H_{\rm T}}{\partial \lambda} = \left[-\left(\sigma_k^2 - 1\right)\right] \tag{3.7d}$$

$$\partial_+ u = \frac{\partial H_{\rm T}}{\partial \Pi_u} = 0 \tag{3.7e}$$

$$-\partial_+ \Pi_u = \frac{\partial H_{\rm T}}{\partial u} = p_\lambda \tag{3.7f}$$

$$\partial_+ v = \frac{\partial H_{\rm T}}{\partial \Pi_v} = 0 \tag{3.7g}$$

$$-\partial_+ \Pi_v = \frac{\partial H_{\rm T}}{\partial v} = [\Pi_k - \partial_- \sigma_k] \tag{3.7h}$$

These are the equations of motion that preserve the constraints of the theory χ_1 and χ_2 in the course of time. For the equal LC time $(x^+ = y^+)$ Poisson bracket $\{,\}_{\rm p}$ of two functions A and B, we choose the convention

$$\{A(x), B(y)\}_{\rm p} := \int dz^- \sum_\alpha \left[\frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)}\right] \tag{3.8}$$

demanding that primary constraint χ_1 be preserved in the course of time, and we obtain the secondary Gauss law constraint of the theory as

$$\chi_3 := \{\chi_1, \mathscr{H}_{\rm T}\}_{\rm p} = \left[\sigma_k^2 - 1\right] \approx 0. \tag{3.9}$$

Now the preservation of χ_2 and χ_3 for all time does not give rise to any further constraints. The theory is thus seen to possess a set of three constraints χ_i $(i = 1, 2, 3)$:

$$\chi_1 = p_\lambda \approx 0 \tag{3.10a}$$

$$\chi_2 = [\Pi_k - \partial_- \sigma_k] \approx 0 \tag{3.10b}$$

$$\chi_3 = \left[\sigma_k^2 - 1\right] \approx 0 \tag{3.10c}$$

The matrix of the Poisson brackets of the constraints χ_1, namely, $S_{\alpha\beta}(w^-, z^-) := \{\chi_\alpha(w^-), \chi_\beta(z^-)\}_\mathrm{p}$, is then calculated. The nonvanishing matrix elements of the matrix $S_{\alpha\beta}(w^-, z^-)$ (with the arguments of the field variables being suppressed) are

$$S_{22} = [-2\partial_- \delta(w^- - z^-)] \tag{3.11a}$$

$$S_{23} = -S_{32} = [-2\,\sigma_k\,\delta(w^- - z^-)] \tag{3.11b}$$

The inverse of the matrix $S_{\alpha\beta}$ does not exist and therefore the matrix is singular, implying that the set of constraints χ_i is first-class and that the theory is a GI theory (Mitra and Rajaraman, 1990a,b). In fact, the action of theory is seen to be invariant under the local vector gauge transformation (LVGT)

$$\delta\,\sigma_k = \beta \qquad \delta\Pi_k = \partial_-\beta \qquad \delta v = \delta_+\beta \tag{3.12a}$$

$$\delta\lambda = \delta u = \delta_{p_\lambda} = \delta\Pi_u = \delta\Pi_v = 0 \tag{3.12b}$$

where $\beta \equiv \beta(x^-, x^+)$ is an arbitrary function of its arguments.

The generator of the above LVGT is the charge operator of the theory

$$J^+ = \int j^+\,dx^- = \int dx^-[\beta(\partial_-\sigma_k)] \tag{3.13}$$

The current operator of the theory is

$$J^- = \int j^-\,dx^- = \int dx^-[\beta(\partial_+\sigma_k)] \tag{3.14}$$

The divergence of the vector-current density, namely, $\partial_\mu j^\mu$ $(=\partial_+ j^+ + \partial_- j^-)$, is therefore seen to vanish. This implies that the theory possesses at the classical level, a local vector gauge symmetry. We now proceed to quantize the theory under the gauge

$$\mathscr{G} = \lambda = 0 \tag{3.15}$$

Under this gauge, the total set of constraints of the theory becomes

$$\psi_1 = \chi_1 = p_\lambda \approx 0 \tag{3.16a}$$

$$\psi_2 = \chi_2 = [\Pi_k - \partial_-\sigma_k] \approx 0 \tag{3.16b}$$

$$\psi_3 = \chi_3 = \left[\sigma_k^2 - 1\right] \approx 0 \tag{3.16c}$$

$$\psi_4 = \mathscr{G} = \lambda = 0 \tag{3.16d}$$

The matrix of the Poisson brackets of the constraints ψ_i, namely, $T_{\alpha\beta}(w, z) := \{\psi_\alpha(w), \psi_\beta(z)\}_\mathrm{p}$, is then calculated. The nonvanishing matrix elements of the matrix $T_{\alpha\beta}$ (w, z) (with the arguments of the field variables being suppressed again) are

$$T_{14} = -T_{41} = -\delta(w^- - z^-) \tag{3.17a}$$

$$T_{22} = -2\partial_-\delta(w^- - z^-) \tag{3.17b}$$

$$T_{23} = -T_{32} = -2\sigma_k\,\delta(w^- - z^-) \tag{3.17c}$$

The inverse of the matrix $T_{\alpha\beta}$ exists and the matrix is nonsingular. The nonvanishing elements of the inverse of the matrix $T_{\alpha\beta}$, i.e., the elements of the matrix $(T^{-1})_{\alpha\beta}$ (with the arguments of the field variables being suppressed once again), are

$$(T^{-1})_{14} = -(T^{-1})_{41} = \delta(w^- - z^-) \tag{3.18a}$$

$$(T^{-1})_{23} = -(T^{-1})_{32} = \left[\frac{1}{2\sigma_k}\right]\delta(w^- - z^-) \tag{3.18b}$$

$$(T^{-1})_{33} = \left[\frac{1}{2\sigma_k^2}\right]\partial_-\delta(w^- - z^-) \tag{3.18c}$$

with

$$\int dz^-\, T(x^-, z^-)\, T^{-1}(z^-, y^-) = 1_{4\times 4}\,\delta(x^- - y^-) \tag{3.19}$$

and

$$[\|\det(T_{\alpha\beta})\|]^{1/2} = 2\sigma_k\,\delta(w^- - z^-) \tag{3.20}$$

Now following the Dirac quantization procedure in the Hamiltonian formulation, one finds that there do not exist any nonvanishing equal LC time commutators for this theory under the gauge $\lambda = 0$. The same is seen to hold true for the quantization of the theory under some other gauge-fixing conditions such as $(\lambda - \sigma_k) = 0$, $(\lambda - \Pi_k) = 0$, and $(\lambda - \sigma_k - \Pi_k) = 0$. This is an interesting result to be noted here and its consequences need to be studied further involving the methods of constraint quantization. The path integral quantization of this theory is, however, possible as usual under all the above gauge-fixing conditions. In the following, we illustrate the path integral quantization of this theory under the gauge $\lambda = 0$, as an example. Also, for later use (in the next section), for considering the BRST formulation of our GI theory, we convert the total Hamiltonian density $\mathscr{H}_{\mathrm{T}}$ into the first-order Lagrangian density

$$\mathscr{L}_{\mathrm{IO}} := [\Pi_k(\partial_+\sigma_k) + p_\lambda(\partial_+\lambda) + \Pi_u(\partial_+u) + \Pi_v(\partial_+v) - \mathscr{H}_{\mathrm{T}}] \tag{3.21a}$$

$$= \left[\lambda\left(\sigma_k^2 - 1\right) + (\partial_-\sigma_k)(\partial_+\sigma_k) + \Pi_u(\partial_+u) + \Pi_v(\partial_+v)\right] \tag{3.21b}$$

In the above equation the terms $p_\lambda(\partial_+\lambda - u)$ and $\Pi_k(\partial_+\sigma_k - v)$ drop out in view of the Hamiltons equations of the theory.

The transition to quantum theory in the path integral formulation is made by writing the vacuum-to-vacuum transition amplitude called the generating functional $z[J_i]$ in the presence of external source currents J_i under the gauge $\zeta = \lambda \approx 0$ as (Henneaux and Teitelboim, 1992; Nemeschansky *et al.*, 1988)

$$Z[J_i] = \int [d\mu] \exp\left[i \int dx^+ dx^- \left[J_i\phi^1 + \lambda\left(\sigma_k^2 - 1\right) + (\partial_-\sigma_k)(\partial_+\sigma_k) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v)\right]\right. \tag{3.22a}$$

where ϕ^i are the phase space variables

$$\phi^i \equiv (\sigma_k, \lambda, u, v) \tag{3.22b}$$

and the functional measure $[d\mu]$ for the above generating functional is

$$\begin{aligned} [d\mu] = [2, \sigma_k\delta(x^- - y^-)][d\sigma_k][d\Pi_k][d\lambda][dp_\lambda][du] \\ [d\Pi_u][dv][d\Pi_v]\delta[(p\lambda)] \approx 0]\delta[(\Pi_k - \partial_-\sigma_k) \approx 0] \\ \delta\left[\left(\sigma_k^2 - 1\right) \approx 0\right]\delta[(\lambda) \approx 0] \end{aligned} \tag{3.22c}$$

4. THE BRST FORMULATION

We now rewrite our GNLSM which is GI as a quantum system that possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our GI GNLSM and replace the notion of gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics. We then introduce new anticommuting variables c and $\bar{c}$ (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable b (called the Nakamishi–Lautrup field) such that (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1998; Tyutini, 1975):

$$\hat{\delta}\sigma_k = c \qquad \hat{\delta}\Pi_k = \partial_- c \qquad \hat{\delta}v = \partial_+ c \tag{4.1a}$$

$$\hat{\delta}\lambda = \hat{\delta}u = \hat{\delta}\Pi_u = \hat{\delta}\Pi_v = \hat{\delta}p_\lambda = 0 \tag{4.1b}$$

$$\hat{\delta}c = 0 \qquad \hat{\delta}\bar{c} = b \qquad \hat{\delta}b = 0 \tag{4.1c}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical variables to be a function $f(\Pi_k, p_\lambda, \Pi_u, \Pi_v, p_b, \Pi_c, \Pi_{\bar{c}}, \sigma_k, \lambda, u, v, b, c, \bar{c})$ such that $\hat{\delta} f = 0$.

4.1. Gauge Fixing in the BRST Formulism

Permorming gauge fixing in the BRST formalism implies adding to the first-order Lagrangian density $\mathscr{L}_{\rm IO}$, a trivial BRST-invariant function (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1998; Tyutini, 1975). We thus write

$$\mathscr{L}_{\rm BRST} = \Big[\lambda\big(\sigma_k^2 - 1\big) + (\partial_-\sigma_k)(\partial_+\sigma_k) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) \\ + \hat{\delta}\Big[\overline{c}(\dot{\partial}_+\partial_+\sigma_k + \partial_+\lambda - \frac{1}{2}b\Big] \tag{4.2}$$

The last term in the above equation is the extra BRST-invariant gauge-fixing term. After one integration by parts, the above equation can now be written as

$$\mathscr{L}_{\rm BRST} = \Big[\lambda\big(\sigma_k^2 - 1\big) + (\partial_-\sigma_k)(\partial_+\sigma_k) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) \\ + b(\partial_+\partial_+\sigma_k + \partial_+\lambda) - \frac{1}{2}b^2 + (\partial_+\overline{c})(\partial_+ c)\Big] \tag{4.3}$$

Proceeding classically, the Euler–Lagrange equation for b reads

$$-b = [\partial_+\partial_+\sigma_k + \partial_+\lambda] \tag{4.4}$$

The requirement $\hat{\delta}b = 0$ then implies

$$-\hat{\delta}b = [\hat{\delta}\,\partial_+\partial_+\sigma_k + \hat{\delta}\partial_+\lambda] \tag{4.5}$$

which in turn implies

$$\partial_+\partial_+ c = 0 \tag{4.6}$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathscr{L}_{\rm BRST}$ with respect to $\overline{c}$. In introducing momenta one has to be careful in defining those for the fermionic variables. We thus define the bosonic momenta in the usual manner so that

$$p_\lambda := \frac{\partial}{\partial(\partial_+\lambda)}\mathscr{L}_{\rm BRST} = b \tag{4.7}$$

but for the fermionic momenta with directional derivatives we set

$$\Pi_c = \mathscr{L}_{\rm BRST}\frac{\overleftarrow{\partial}}{\partial(\partial_+ c)} = (\partial_+\overline{c}) \qquad \Pi_{\overline{c}} = \frac{\overrightarrow{\partial}}{\partial(\partial_+\overline{c})}\mathscr{L}_{\rm BRST} = (\partial_+ c) \tag{4.8}$$

implying that the variable canonically conjugate to c is $(\partial_+\overline{c})$ and the variable conjugate to $\overline{c}$ is $(\partial_+ c)$. For writing the Hamiltonian density from the Lagrangian density in the usual manner we remember that the former has to be Hermitian so that

$$\mathscr{H}_{\rm BRST} = [\Pi_k(\partial_+\sigma_k) + p_\lambda(\partial_+\lambda) + \Pi_u(\partial_+u) + \Pi_v(\partial_+v) + \Pi_c(\partial_+c) + (\partial_+\bar{c})\Pi_{\bar{c}} - \mathscr{L}_{\rm BRST}] \tag{4.9a}$$

$$= \left[-p_\lambda(\partial_+\partial_+\sigma_k - \partial_+\lambda) - \frac{1}{2}(p\lambda)^2 - \lambda\left(\sigma_k^2 - 1\right) + \Pi_c\Pi_{\bar{c}} \right] \tag{4.9b}$$

We can check the consistency of (4.8) and (4.9) by looking at Hamilton's equations for the fermionic variables, i.e.,

$$\partial_+c = \frac{\overrightarrow{\partial}}{\partial\Pi_c}\,\mathscr{H}_{\rm BRST} \qquad \partial_+\bar{c} = \mathscr{H}_{\rm BRST}\frac{\overleftarrow{\partial}}{\partial\Pi_{\bar{c}}} \tag{4.10}$$

Thus we see that

$$\partial_+c = \frac{\overrightarrow{\partial}}{\partial\Pi_c}\,\mathscr{H}_{\rm BRST} = \Pi_{\bar{c}} \qquad \partial_+\bar{c} = \mathscr{H}_{\rm BRST}\frac{\overleftarrow{\partial}}{\partial\Pi_{\bar{c}}} = \Pi_c \tag{4.11}$$

is in agreement with (4.8). For the operators c, $\bar{c}$, ∂_+c and $\partial_+\bar{c}$, one needs to satisfy the anticommutation relations of ∂_+c with $\bar{c}$ or of $\partial_+\bar{c}$ with c, but not of c with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0 \qquad \partial_+\{\bar{c}, c\} = 0 \tag{4.12a}$$

$$\{\partial_+\bar{c}, c\} = (-1)\{\partial_+c, \bar{c}\} \tag{4.12b}$$

where {,} means an anticommutator. We thus see that the anticommulators in (4.12b) are nontrivial and need to be fixed. In order to fix these, we demand that c satisfy the Heisenberg equation (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1998; Tyutini, 1975):

$$[c,\ \mathscr{H}_{\rm BRST}] = i\ \partial_+c \tag{4.13}$$

and using the property $c^2 = c^{-2} = 0$ one obtains

$$[c,\ \mathscr{H}_{\rm BRST}] = \{\partial_+\bar{c}, c\}\partial_+c \tag{4.14}$$

Equations (4.12)–(4.14) then imply

$$\{\partial_+\bar{c}, c\} = (-1)\{\partial_+c, \bar{c}\} = i \tag{4.15}$$

Here the minus sign in the above equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1998; Tyutini, 1975).

4.2. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformations (4.1). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators that satisfy Bose and fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy

$$[\sigma_k, Q] = \partial_+ c \qquad [\Pi_k, Q] = [2c\,\sigma_k - \partial_-\partial_+ c] \qquad [\lambda, Q] = \partial_+ c \tag{4.16a}$$

$$\{\bar{c}, Q\} = [-\partial_-\sigma_k + p_\lambda + \Pi_k] \qquad \{\partial_+\bar{c}, Q\} = (-1)\big[\sigma_k^2 - 1\big] \tag{4.16b}$$

All other commutators and anticommutators involving Q vanish. In view of (4.16), the BRST charge operator of the present theory can be written as

$$Q = \int dx^-\big[ic\big[\sigma_k^2 - 1\big] - i(\partial_+ c)[p_\lambda + \Pi - \partial_-\sigma_k]\big] \tag{4.17}$$

This equation implies that the set of states satisfying the conditions

$$p_\lambda|\psi\rangle = 0 \tag{4.18a}$$

$$[\Pi_k - \partial_-\sigma_k]|\psi\rangle = 0 \tag{4.18b}$$

$$\big[\sigma_k^2 - 1\big]|\psi\rangle = 0 \tag{4.18c}$$

belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belong to the set of BRST-invariant states.

In order to understand the condition needed for recovering the physical states of the theory we rewrite the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider (4.6). The solution of this equation gives (for the LC time $x^+ \equiv t$) the Heisenberg operator $c(t)$ (and correspondingly $\bar{c}(t)$) as

$$c(t) = Gt + F \qquad \bar{c}(t) = G^\dagger t + F^\dagger \tag{4.19}$$

which at time $t = 0$ imply

$$c \equiv c(0) = F \qquad \bar{c} \equiv \bar{c}(0) = F^\dagger \tag{4.20a}$$

$$\partial_+ c \equiv \partial_+ c(0) = G \qquad \partial_+\bar{c} \equiv \partial_+\bar{c}(0) = G^\dagger \tag{4.20b}$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\partial_+\bar{c}, \partial_+ c\} = 0 \tag{4.21a}$$

$$\{\partial_+\bar{c}, c\} = i = -\{\partial_+ c, \bar{c}\} \tag{4.21b}$$

we then obtain

$$F^2 = F^{\dagger 2} = \{F^\dagger, F\} = \{G^\dagger, G\} = 0 \tag{4.22a}$$

$$\{G^\dagger, F\} = i \qquad \{G, F^\dagger\} = -i \tag{4.22b}$$

We now let $|0\rangle$ denote the fermionic vacuum for which

$$G|0\rangle = F|0\rangle = 0 \tag{4.23}$$

Defining $|0\rangle$ to have norm one, (4.22b) implies

$$\langle 0|FG^\dagger|0\rangle = i \qquad \langle 0|GF^\dagger|0\rangle = -i \tag{4.24}$$

so that

$$G^\dagger|0\rangle \neq 0 \qquad F^\dagger|0\rangle \neq 0 \tag{4.25}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathscr{H}_{\text{BRST}}$ is however irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators

$$\mathscr{H}_{\text{BRST}} = \left[-p_\lambda(\partial_+\partial_+\sigma_k - \partial_+\lambda) - \frac{1}{2}(p_\lambda)^2 - \lambda\left(\sigma_k^2 - 1\right) + G^\dagger G\right] \tag{4.26}$$

and the BRST charge operator Q is

$$Q = \int dx^-\left[iF\left(\sigma_k^2 - 1\right) - iG(p_\lambda + \Pi_k - \partial_-\sigma_k)\right] \tag{4.27}$$

Now because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (4.18) hold but also additional states for which

$$B|\psi\rangle = D|\psi\rangle = 0 \tag{4.28a}$$

$$p_\lambda|\psi\rangle \neq 0 \tag{4.28b}$$

$$[\Pi_k - \partial_-\sigma_k]|\psi\rangle \neq 0 \tag{4.28c}$$

$$\left[\sigma_k^2 - 1\right]|\psi\rangle \neq 0 \tag{4.28d}$$

The Hamiltonian is also invariant under the anti-BRST transformation given by

$$\hat{\bar{\delta}}\sigma_k = -\bar{c} \qquad \hat{\bar{\delta}}\Pi_k = -\partial_-\bar{c} \qquad \hat{\bar{\delta}}v = -\partial_+\bar{c} \tag{4.29a}$$

$$\hat{\bar{\delta}}\lambda = \hat{\bar{\delta}}u = \hat{\bar{\delta}}p_\lambda = \hat{\bar{\delta}}\Pi_u = \hat{\bar{\delta}}\Pi_v = 0 \tag{4.29b}$$

$$\hat{\bar{\delta}}\bar{c} = \qquad \hat{\bar{\delta}}c = -b \qquad \hat{\bar{\delta}}b = 0 \tag{4.29c}$$

with the generator or anti-BRST charge

$$\bar{Q} = \int dx^{-}\left[-i\bar{c}\left[\sigma_k^2 - 1\right] + i(\partial_{+}\bar{c})[p_\lambda + \Pi_k - \partial_{-}\sigma_k]\right] \tag{4.30a}$$

$$= \int dx^{-}\left[-iF^{\dagger}\left(\sigma_k^2 - 1\right) + iG^{\dagger}(p_\lambda + \Pi_k - \partial_{-}\sigma_k)\right] \tag{4.30b}$$

We also have

$$\partial_{+}Q = [Q, H_{\mathrm{BRST}}] = 0 \tag{4.31a}$$

$$\partial_{+}\bar{Q} = [\bar{Q}, H_{\mathrm{BRST}}] = 0 \tag{4.31b}$$

with

$$H_{\mathrm{BRST}} = \int dx\ \mathscr{H}_{\mathrm{BRST}} \tag{4.31c}$$

and we further impose the dual condition that both Q and $\bar{Q}$ annihilate physical states, implying that

$$Q|\psi\rangle = 0 \quad \text{and} \quad \bar{Q}|\psi\rangle = 0 \tag{4.32}$$

The states for which (4.18) hold, satisfy both of these conditions and, in fact, are the only states satisfying both of these conditions, since, although with (4.22)

$$G^{\dagger}G = -GG^{\dagger} \tag{4.33}$$

there are no states of this operator with $G^{\dagger}|0\rangle = 0$ and $F^{\dagger}|0\rangle = 0$ [cf. (4.25)], and hence no free eigenstates of the fermionic part of H_{BRST} that are annihilated by each of G, $G^{\dagger}$, F, $F^{\dagger}$. Thus the only states satisfying (4.32) are those satisfying the constraints of the theory.

Further, the states for which (4.18) hold satisfy both the conditions (4.32) and infact, are the only states satisfying both of these conditions because in view of (4.21) one cannot have simultaneously c, $\partial_{+}c$ and $\bar{c}$, $\partial_{+}\bar{c}$, applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (4.32) are those that satisfy the constraints of the theory and they belong to the set of BRST-invariant and anti-BRST-invariant states.

Alternatively, one can understand the above point in terms of fermionic annihilation and creation operators as follows. The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which (4.18) hold but also additional states for which (4.28) hold. However $\bar{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\bar{Q}$ contains only the states for which (4.18) hold, simply because $G^{\dagger}|\psi\rangle \neq 0$ and $F^{\dagger}|\psi\rangle \neq 0$. Thus in this alternative way also we see that the states satisfying $Q|\psi\rangle = \bar{Q}|\psi\rangle = 0$ (i.e., satisfying (4.32)) are only those states that satisfy the constraints of the theory and also that these states belong to the set of BRST-invariant and anti-BRST-invariant states.

5. SUMMARY AND DISCUSSIONS

In this work we have studied the NLSM in the FF, i.e., on the hyperplanes $x^+ = (x^0 + x^1)/\sqrt{2} =$ constant. The theory in the IF has been studied before rather widely (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b) and it is well known to be a GNI theory possessing a set of four second-class constraints (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b).

The FF theory on the other hand as studied in the present case is seen to possess a set of three first-class constraints and consequently it describes a GI theory. Also for the FF theory studied in the present work there does not exist any problem with respect to operator ordering as one encounters in the case of the IF theory (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b).

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